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Contributed Paper

ADAPTIVE EXPONENTIAL TRACKING FOR NONLINEARLY PERTURBED MINIMUM PHASE SYSTEMS*

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Abstract. The class of systems under consideration consists of multi-input, multi-output, finite dimensional, state space systems subject to nonlinearities in the input-, state- and output-variables and of unmodeled systems dynamics. The systems and its state dimension are not known precisely. However, structural information is assumed, such as the linear system is minimum phase, the spectrum of the high-frequency gain matrix lies either in the open right- or left-half plane. For different classes of systems, simple adaptive high-gain stabilizers — not based on identification or estimation algorithms — are presented, which, in the presence of certain nonlinearities, ensure exponential decay of the motion of the closed-loop system and finite gain convergence of the parameters of the adaptive controllers. Using these results in cooperation with an internal model, an adaptive tracking controller, which guarantees exponential decay of the error between the output and reference signals belonging to a known solution space of a differential equation, is presented for linear systems.

Key Words—Adaptive control, adaptive stabilization, adaptive tracking, robust control, feedback control, nonlinear control.

1. Introduction

The area of adaptive stabilization and tracking is one of current research interest with a number of publications in the field mainly centered around existence results and the proofs of asymptotic and L_p-stability for a number of different algorithms. The central problem considered is the construction of nonlinear measurement feedback controllers of simple structure, capable of stabilizing all systems in a specified class independent of a large degree of ignorance of the details of the system's dynamical behavior. Two ingredients of the approach which distinguish it from other adaptive control approaches are that
(1) No attempt is made to identify systems dynamics (i.e., the controller remains completely ignorant of the system that it is controlling, despite its success in controlling that system) and
(2) the class of systems considered is specified by a definition of system structure rather than by specification of system parameters or state dimension. In

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fact, in many approaches, the state dimension can be arbitrary, leading to the use of the suggestive name of Universal Adaptive Stabilizers resp. Tracking Controllers for the controllers considered.

One of the basic problems of universal adaptive stabilization has centered around a class of multi-input, multi-output, minimum phase, relative degree one systems. More precisely systems of the familiar form

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) \in \mathbb{R}^n \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{1.1}$$

in the class Σ consisting of linear systems $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ which satisfy the ***minimum phase*** condition

$$\det \begin{bmatrix} sI-A & B \\ -C & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \overline{\mathbb{C}}_+, \tag{1.2}$$

and also possess a ***high-frequency gain matrix*** with spectrum either in the left or right half plane, i.e.,

$$\sigma(CB) \subset \mathbb{C}_- \quad \text{or} \quad \sigma(CB) \subset \mathbb{C}_+, \tag{1.3}$$

but unknown in which half plane.

Note, that a system lies in Σ does not require that either A, B, C or the state dimension n be known.

The first *simple* high-gain adaptive stabilizer for linear single-input, single-output systems was introduced by Willems and Byrnes (1984) and can be described as follows. The output feedback law

$$u(t) = -S(k(t))k(t)y(t) \tag{1.4}$$

depends only on the output measurements and a monotonically non-decreasing gain adaptation produced by

$$\dot{k}(t) = y(t)^2, \quad k(0) \in \mathbb{R}. \tag{1.5}$$

(1.4), (1.5) applied to any system $(A, B, C) \in \Sigma$ produces a closed-loop system

$$\begin{aligned} \dot{x}(t) &= [A - S(k(t))k(t)BC]x(t), \quad x(0) \in \mathbb{R}^n, \\ y(t) &= Cx(t), \\ \dot{k}(t) &= y(t)^2, \quad k(0) \in \mathbb{R} \end{aligned}$$

with $\lim_{t\to\infty} x(t) = 0$ and finite $\lim_{t\to\infty} k(t)$. If CB is known to be positive, then $S(\cdot)$ in (1.4) is chosen $S(\cdot) \equiv 1$. For unknown high-frequency gain $CB \neq 0$, Nussbaum (1983) has introduced so called switching functions in order to search the "correct" sign, a simple example due to Willems and Byrnes is $S(k(t)) = \sin\sqrt{|k(t)|}$.

Various authors have investigated the question of how this simple high-gain adaptive control scheme carries over to different classes of systems to be stabilized or tracked (see Morse, 1985; Mårtensson, 1986; Owens et al., 1987; Ryan, 1988; 1991; Mudgett and Morse, 1989; Logemann, 1990; Logemann and Ilchmann, 1991; Ilchmann and Logemann, 1992), to name but a few. The present paper is in the same spirit. Compared to previous results, we present the following extensions:

(i) a whole *class* of exponential stabilizers is presented, which contains the simple Willems-Byrnes stabilizer as an example,
(ii) an exponential weighting of the gain adaptation mechanisms ensures *exponential* decay of the state of the closed-loop system,
(iii) exponential stability is retained, even if $(A, B, C) \in \Sigma$ is subjected to certain (small) *nonlinearities* in the state space and arbitrary large gain bounded nonlinearities in the input, combined with nonlinear memoryless input and output (i.e., actuator and sensor) characteristics,
(iv) L_p*-based* adaptation mechanism for $p \geq 1$ are developed, which shows that common L_2 based methods are only a special case,
(v) a previously used L_2-inequality for linear systems is extended to an L_p*-inequality* including exponential weighting of the signals and nonlinear systems perturbations, so that the present output can be related to past input and output values only,
(vi) *exponential tracking* for certain reference signals can be achieved if previous results are combined with an internal model, and the system is linear,
(vii) all previous results hold true in the *multivariable* case, where the sign of the high-frequency gain is unknown.

The form of system to be considered is hence of the basic form (1.1) perturbed by nonlinear terms. More precisely, let $p \geq 1$, and assume that the maps

$$\left.\begin{array}{l} d : [0, \infty) \to \mathbb{R}^n \\ \quad \exp(\varepsilon t) d(t) \in L_p(0, \infty) \quad \text{for some } \varepsilon > 0 \end{array}\right\}, \tag{1.6}$$

$$\left.\begin{array}{l} g : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n, \quad (t, x) \mapsto g(t, x) \\ \quad \|g(t, x)\| \leq \hat{g} \|x\| \quad \text{for almost all } t \geq 0 \text{ and all } x \in \mathbb{R}^n \end{array}\right\}, \tag{1.7}$$

$$\left.\begin{array}{l} h : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^m, \quad (t, x) \mapsto h(t, x) \\ \quad \|h(t, x)\| \leq \hat{h} \|x\| \quad \text{for almost all } t \geq 0 \text{ and all } x \in \mathbb{R}^n \end{array}\right\}, \tag{1.8}$$

$$\xi : [0, \infty) \times \mathbb{R} \to \mathbb{R}^m, \quad (t, \tilde{u}) \mapsto \xi(t, \tilde{u}), \tag{1.9}$$

$$\eta : [0, \infty) \times \mathbb{R} \to \mathbb{R}^m, \quad (t, y) \mapsto \eta(t, y) \tag{1.10}$$

be *Carathéodory functions*†, and satisfy the above bounds for some *unknown* $\hat{g}$, $\hat{h} \geq 0$. These functions are incorporated into the underlying linear system (1.1) in the form

† $f : [0, \infty) \times \mathbb{R}^q \to \mathbb{R}$ is called a Carathéodory function, if $f(\cdot, x) : t \mapsto f(t, x)$ is measurable on $[0, \infty)$ for each $x \in \mathbb{R}^q$, and $f(t, \cdot) : x \mapsto f(t, x)$ is continuous on $\mathbb{R}^q$ for all $t \geq 0$.

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + g(t, x(t)) + d(t) \\ &\qquad + B[u(t)+h(t, x(t))], \qquad x(0) \in \mathbb{R}^n \\ y(t) &= Cx(t) \\ u(t) &= \xi(t, \tilde{u}(t)), \quad \tilde{y}(t) = \eta(t, y(t)) \end{aligned}\right\} \tag{1.11}$$

to create the nonlinear system to be controlled (see Fig. 1), and note the interpretation of ξ and η as memoryless nonlinear input and output characteristics.

It follows from the theory of ordinary differential equations, that, since g, d, h are Carathéodory functions, the initial value problem (1.11) has a solution with maximal interval of existence $[0, t')$, where $t' \in (0, \infty]$. $x(\cdot) : [0, t') \to \mathbb{R}^n$ is called a solution of (1.11) if it is absolutely continuous on compact intervals and satisfies (1.11) for almost all $t \in [0, t')$. If, in addition, it is assumed that g is locally Lipschitz in x for each fixed t, and g, h, u are locally integrable on t for each fixed x, then the initial value problem has a ***unique*** solution.

The term $d(\cdot)$ can be regarded as an arbitrary L_p-system disturbance of the state whilst $g(\cdot, \cdot)$ is a state nonlinearity. The possibility of including $g(\cdot, \cdot)$ shows well-posedness of the problem, since all results hold true if the linear bound $\hat{g}$ is small enough.

The function $h(\cdot, \cdot)$ reflects some unmodeled feedback loops in the system. If $\xi(\cdot, \cdot)$, $\eta(\cdot, \cdot)$ are linearly sector bounded, then the general conclusion of the paper is that the feedback law generating the available real input $\tilde{u}(t)$ from the available real measurement $\tilde{y}(t)$ via the relation

$$\tilde{u}(t) = -S(t)k(t)\tilde{y}(t) \tag{1.12}$$

is capable of ensuring ***exponential decay*** of the solution of the closed-loop system

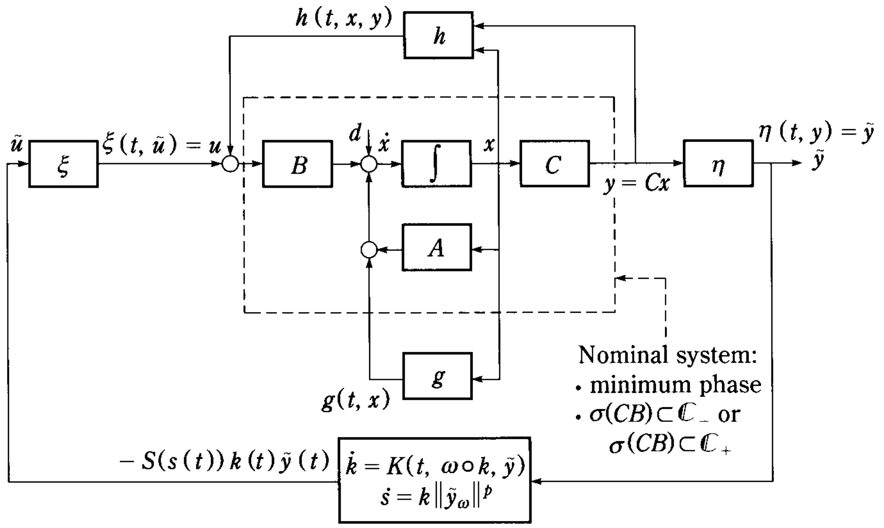

Fig. 1. Closed-loop adaptive system.

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + B\xi(t, -S(t)k(t)\eta(t, y(t))) \\ &\qquad + g(t, x(t)) + Bh(t, x(t), y(t)) \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{1.13}$$

by choice of a switching function $S(t)$ and a gain evolution $k(t)$ dependent upon exponentially weighted versions of the output responses. The results considerably strengthen previous results which are concerned only with asymptotic stabilization of linear systems. If the linear system (1.1) is not subjected to any nonlinearities, but unknown signals $d(\cdot)\in L_p(0, \infty)$ are allowed in the state space, then previous exponential adaptive stabilizers in combination with an internal model are exponential adaptive tracking controllers for certain classes of reference signals.

In Sec. 2, basic properties of the system class Σ, i.e., systems of the form (1.1) satisfying (1.2) and (1.3), are proved. While the proofs of these results are rather technical in nature, much of the analysis centers upon the proof of and use of an integral inequality description of exponentially weighted system input/output dynamics, the results provide structural clarity in the stability proofs of the adaptive controllers presented in the following sections. Section 3 concentrates on the proof of exponential stabilization in the case of nonlinear perturbations of systems in Σ, with the extra assumption that the sign of CB is known. Section 4 extends these results to include the situation where the sign of CB is not known by the use of switching functions of Nussbaum form satisfying a scaling invariance property, originally introduced by Logemann and Owens (1988). In Sec. 5, the previous results in combination with an internal model are used to design an adaptive tracking controller for linear systems, which ensures exponential decay of the tracking error. Finally, in Sec. 6, we illustrate the results of Theorem 4.6 by the simulations of two examples.

Nomenclature.

$\mathbb{C}_+ (\mathbb{C}_-)$: open right- (left-) half complex plane.

$\| x \|_P = \sqrt{<x, Px>}$ for $x \in \mathbb{R}^n$, $P = P^T \in \mathbb{R}^{n\times n}$ positive, $\| x \| := \| x \|_{I_n}$.

$$\operatorname{sgn}(X) = \begin{cases} +1, & \text{if } \sigma(X) \subset \mathbb{C}_+, \\ -1, & \text{if } \sigma(X) \subset \mathbb{C}_-. \end{cases}$$

2. Properties of the Systems Class

The proofs of the results in the following sections depend crucially on a number of technical details associated with the structure of the problem and the need to use descriptions that contain only input and output data and yet enable the derivation of results on systems state space behavior. Some of the results are fundamental properties of the class Σ. The purpose is, to obtain a deeper understanding of the systems class and to derive consequences which will be used to clarify the stability proofs in the following sections.

To obtain an explicit characterization of the minimum phase condition (1.2) the following definition is needed.

Suppose $G(s)\in \mathbb{R}(s)^{m\times m}$ is a rational matrix with Smith-McMillan form

$$\operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \cdots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0, \cdots, 0\right) = U(s)^{-1}G(s)V(s)^{-1}, \tag{2.1}$$

where $U(s)$, $V(s) \in \mathbb{R}[s]^{m\times m}$ are unimodular, $rk_{\mathbb{R}(s)}\, G(\cdot) = r$, $\varepsilon_i(s) | \varepsilon_{i+1}(s)$, $\psi_{i+1}(s) | \psi_i(s)$ and $\varepsilon_i(s)$, $\psi_i(s) \in \mathbb{R}[s]$ are coprime for $i = 1, \cdots, r$. Set

$$\varepsilon(s) := \prod_{i=1}^{r} \varepsilon_i(s), \quad \psi(s) := \prod_{i=1}^{r} \psi_i(s).$$

Then, s_0 is called a *zero* (respectively, a *pole* of $G(s)$), if $\varepsilon(s_0) = 0$ (respectively, $\psi(s_0) = 0$).

Now we are in a position to characterize condition (1.2) and show that it is a multivariable extension of the minimum phase definition given usually in the frequency domain for single-input single-output systems, provided they are stabilizable and detectable.

2.1 Proposition The system (A, B, C) of the form (1.1) is minimum phase, i.e., it satisfies (1.2), if and only if it fulfils the conditions

(i) $rk[sI_n - A,\ B] = n$ for all $s \in \overline{\mathbb{C}}_+$, i.e., (A, B) is stabilizable by state feedback,

(ii) $rk \begin{bmatrix} sI_n - A \\ C \end{bmatrix} = n$ for all $s \in \overline{\mathbb{C}}_+$, i.e., (A, C) is detectable,

(iii) $C(sI_n - A)^{-1}B \in \mathbb{R}(s)^{m\times m}$ has no zeros in $\overline{\mathbb{C}}_+$.

Proof. For $G(s) := C(sI_n - A)^{-1}B$ we use the decomposition (2.1). Coppel (1974), Theorem 10, has proved that, if (A, B, C) is detectable and stabilizable, then $s_0 \in \overline{\mathbb{C}}_+$ is a zero of $\psi(\cdot)$ (including multiplicity), if and only if it is a zero of $\det(\cdot I_n - A)$. Now the result follows directly from:

$$\begin{vmatrix} sI_n - A & B \\ -C & 0 \end{vmatrix} = |C \operatorname{adj}(sI_n - A)B| = |sI_n - A| \cdot |C(sI_n - A)^{-1}B|$$
$$= \frac{|sI_n - A|}{\psi(s)} \cdot \varepsilon(s).$$

The following lemma provides a convenient system description into which every system with $\det(CB) \neq 0$ can be converted by a suitable state space transformation, representing the direct sum of the state space into the range of B and the kernel of C. In practical terms, it makes possible the separation of the inputs and outputs from the rest of the system states in the remainder of the section by creating a situation, where the remaining states act as a disturbance to output dynamics yet are driven by that output via an internal feedback loop.

2.2 Lemma Consider the system

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n \\ y(t) &= Cx(t) \end{aligned}\right\} \tag{2.2}$$

with $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$.

(i) Suppose $\det(CB) \neq 0$. Let $V \in \mathbb{R}^{n\times(n-m)}$ denote a basis matrix of ker C, then $S := [B(CB)^{-1},\ V]$ has the inverse $S^{-1} = \begin{bmatrix} C \\ T \end{bmatrix}$, where $T := (V^T V)^{-1} V^T [I_n - B(CB)^{-1}C]$. Hence, the state space transformation $\begin{bmatrix} y \\ z \end{bmatrix} = S^{-1}x = \begin{bmatrix} Cx \\ Tx \end{bmatrix}$

converts (2.2) into

$$\left.\begin{aligned}\dot{y}(t) &= A_1y(t) + A_2z(t) + CBu(t)\\ \dot{z}(t) &= A_3y(t) + A_4z(t)\end{aligned}\right. , \quad \left.\begin{bmatrix} y(0)\\ z(0)\end{bmatrix} = S^{-1}x_0\right\}, \tag{2.3}$$

where $A_1\in\mathbb{R}^{m\times m}$, $A_2\in\mathbb{R}^{m\times(n-m)}$, $A_3\in\mathbb{R}^{(n-m)\times m}$, $A_4\in\mathbb{R}^{(n-m)\times(n-m)}$, so that

$$\begin{bmatrix} A_1 & A_2\\ A_3 & A_4\end{bmatrix} = S^{-1}AS.$$

(ii) If $(A,\ B,\ C)\in\Sigma$, i.e., (2.2) satisfies (1.2) and (1.3), then A_4 in (2.3) is asymptotically stable, i.e., $\sigma(A_4)\subset\mathbb{C}_-$.

Proof. (i) is straightforward. (ii) follows from the fact that $\sigma(A_4)\subset\mathbb{C}_-$, and that for all $s\in\overline{\mathbb{C}}_+$ we have

$$\begin{vmatrix} sI_n-A & B\\ -C & 0\end{vmatrix} = \begin{vmatrix} sI_m-A_1 & -A_2 & -CB\\ -A_3 & sI_{n-m}-A_4 & 0\\ I_m & 0 & 0\end{vmatrix} = |-CB|\cdot|sI_{n-m}-A_4| \neq 0.$$

This completes the proof.

In order to include the possibility of exponential stabilization in a similar manner to that of Ilchmann and Owens (1990) we introduce the following notation to describe the creation of an exponentially weighted signal $v_\omega(\cdot)$ from an original $v(\cdot)$. More precisely.

Let $\omega(\cdot):\mathbb{R}\to\mathbb{R}$ be continuously differentiable and $v(\cdot):\mathbb{R}\to\mathbb{R}^r$, $r\in\mathbb{N}$, a vector valued-function. Then $v_\omega(\cdot)$ will be defined by

$$v_\omega(t) := \exp(\omega(t)t)v(t). \tag{2.4}$$

The form of $\omega(t)$ will be important in later sections, but for the moment it can be regarded as arbitrary.

In the following remark, the state space transformation of Lemma 2.2 is applied to the perturbed system (1.11) to produce a representation convenient for the following analysis.

2.3 Remark Suppose $\omega(\cdot):[0,\ \infty)\to[0,\ \infty)$ is continuously differentiable. Consider (1.11) where it is assumed that $(A,\ B,\ C)\in\Sigma$ is subjected to the perturbations (1.6)–(1.8). Then, by using the coordinate transformation given in Lemma 2.2, the new coordinates y_ω, z_ω satisfy

$$\begin{aligned}\dot{y}_\omega(t) &= [A_1+(\omega(t)+\dot{\omega}(t)t)I_m]y_\omega(t) + A_2z_\omega(t) + CBu_\omega(t)\\ &\quad + g_\omega^1(t,y_\omega(t),z_\omega(t)) + Cd_\omega(t) + CBh_\omega(t,y_\omega(t),z_\omega(t)),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\dot{z}_\omega(t) &= A_3y_\omega(t) + [A_4+(\omega(t)+\dot{\omega}(t)t)I_{n-m}]z_\omega(t)\\ &\quad + g_\omega^2(t,y_\omega(t),z_\omega(t)) + Td_\omega(t),\end{aligned} \tag{2.6}$$

where $\sigma(A_4) \subset \mathbb{C}_-$, and

$$g^1(t, \eta, \xi) := Cg(t, \exp(-\omega(t)t)S[\eta^T, \xi^T]^T),$$

$$g^2(t, \eta, \xi) := Tg(t, \exp(-\omega(t)t)S[\eta^T, \xi^T]^T),$$

$$h(t, \eta, \xi) := h(t, \exp(-\omega(t)t)S[\eta^T, \xi^T]^T).$$

(1.7) and (1.8) yield, that for almost all $t \geq 0$, all $(\eta, \xi) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and some $\hat{g}, \hat{h} > 0$, we have

$$\left.\begin{aligned} \| g_\omega^i(t, \eta, \xi) \| &\leq \hat{g} \| C \| \| S \| \| [\eta^T, \xi^T]^T \| \quad \text{for } i = 1, 2 \\ \| h_\omega(t, \eta, \xi) \| &\leq \hat{h} \| S \| \| [\eta^T, \xi^T]^T \| \end{aligned}\right\}. \tag{2.7}$$

That is, the essential structure of the nonlinearities is retained.

The following technical lemma shows a basic consequence of certain properties of L_p-functions of use in the proof of asymptotic stability and boundedness in later sections. This may be well-known, however, we could not find a reference in the literature. Our colleague E.P. Ryan (University of Bath) has proved a similar result (private communication).

2.4 Lemma Suppose $f(\cdot):[0, \infty) \to \mathbb{R}^n$ is any absolutely continuous function satisfying $f(\cdot) \in L_p(0, \infty)$ for some $p \in [1, \infty)$ and $\dot{f}(\cdot) \in L_q(0, \infty)$ for some $q \in [1, \infty]$. Then, $f(\cdot) \in L_i(0, \infty)$ for all $i \in [p, \infty]$, and $\lim_{t \to \infty} f(t) = 0$.

Proof. Put $r = p + 1/q - p$, where $1/q \triangleq 0$ if $q = \infty$ and define

$$J_1 := \{ t \geq 0 \mid f(\cdot) \text{ is not differentiable at } t \},$$

which, by the absolute continuity of $f(\cdot)$, is of measure zero. Set

$$J_2 := \{ t \in [0, \infty) \setminus J_1 \mid f(t) = 0 \text{ and } \dot{f}(t) \neq 0 \}.$$

Now it is easy to see that $\| f(\cdot) \|$ is not differentiable in any point of J_2. However, $\| f(\cdot) \|$ is absolutely continuous because $f(\cdot)$ is, and hence, J_2 must be of measure zero. It follows that $J := J_1 \cup J_2$ is of measure zero and a routine calculation gives:

$$\frac{d}{d\tau} \| f(\tau) \| = \begin{cases} \dfrac{< f(\tau), \dot{f}(\tau) >}{\| f(\tau) \|}, & \tau \in [0, \infty) \setminus J \text{ and } f(\tau) \neq 0, \\ 0, & \tau \in [0, \infty) \setminus J \text{ and } f(\tau) = 0. \end{cases}$$

Since

$$\int_{[0, \infty) \setminus J} \frac{d}{d\tau} \| y(\tau) \|^r d\tau = \int_{[0, \infty)} \frac{d}{d\tau} \| y(\tau) \|^r d\tau,$$

it follows from Hölder's inequality, that for $t \geq t_n \geq 0$ we have

$$
\begin{aligned}
\|f(t)\|^r - \|f(t_n)\|^r &= \int_{t_n}^{t} \frac{d}{d\tau}(\|f(\tau)\|^r)\, d\tau \\
&= r\int_{t_n}^{t} \|f(\tau)\|^{r-1} \cdot \frac{d}{d\tau}(\|f(\tau)\|)\, d\tau \\
&\le r\int_{t_n}^{t} \|f(\tau)\|^{r-1} \cdot \|\dot{f}(\tau)\|\, d\tau \\
&\le r\left[\int_{t_n}^{t} \|f(\tau)\|^{(r-1)\frac{p}{r-1}}\right]^{\frac{r-1}{p}} \cdot \left[\int_{t_n}^{t} \|\dot{f}(\tau)\|^q\, d\tau\right]^{\frac{1}{q}} \\
&= r\|f(\cdot)\|^{r-1}_{L_p(t_n,\, t)} \cdot \|\dot{f}(\cdot)\|_{L_q(t_n,\, t)},
\end{aligned}
$$

and thus, for all $t \ge t_n$

$$
\|f(t)\|^r \le r\|f(\cdot)\|^{r-1}_{L_p(t_n,\, \infty)} \cdot \|\dot{f}(\cdot)\|_{L_q(t_n,\, \infty)} + \|f(t_n)\|^r. \tag{2.8}
$$

Choose a sequence $\{t_n\}_{n\in \mathbb{N}}$ so that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} f(t_n) = 0$. Then,

$$
\lim_{n\to\infty} \|f\|_{L_p(t_n,\, \infty)} = \lim_{n\to\infty} \|\dot{f}\|_{L_q(t_n,\, \infty)} = 0,
$$

and (2.8) proves $\lim_{t\to\infty} f(t) = 0$. Thus, $f(\cdot) \in L_\infty(0, \infty)$ and therefore, $f(\cdot) \in L_p(0, \infty) \cap L_\infty(0, \infty)$. This completes the proof.

The propositions in the remainder of this section represent important steps in a line of argument, which has been quite often used in *high-gain* adaptive control. It is based on a generalization of the following statement for the special situation $p = 2$, $\omega(\cdot) \equiv 0$ and the undisturbed system (1.1). More precisely, suppose (A, B, C) is minimum phase and satisfies $\det(CB) \ne 0$. Then for any positive definite matrix P, the system (1.1) satisfies the following basic inequality:

$$
\frac{1}{2}\|y(t)\|_P^2 \le M + M\int_0^t \|y(\tau)\|^2 d\tau + \int_0^t < y(\tau),\ PCBu(\tau) > d\tau
$$

for some $M > 0$ (see Ilchmann and Owens, 1991). As the control algorithms to be proposed in later sections depend upon exponentially weighted signals, several technical details of the generalization need to be derived and the structure of the argument changes substantially.

We need to prove a generalization of the above inequality for systems in the class Σ. The main results of this paper will use this inequality extensively as it provides a vital link between exponentially weighted input and output dynamic characteristics without the need for an explicit description of the dynamics of the remainder of the state.

2.5 Proposition Suppose $(A, B, C) \in \Sigma$ is subject to the disturbances (1.6), (1.7) in the form (1.11), and $[0, t')$, $t' \in (0, \infty]$, is the maximal interval of an absolutely continuous solution $x(\cdot)$ of (1.11). Suppose furthermore that $\omega(\cdot): [0, t') \to \mathbb{R}$ is continuously differentiable, non-increasing and $\lim_{t\to t'} \omega(t) = 0$. Let $P > 0$, $p \ge 1$ and define

$$\beta : \mathbb{R}^m \to \mathbb{R}^m, \ y \mapsto \beta(y) = \begin{cases} \dfrac{y}{\|y\|_P}, & y \neq 0, \\ 0, & y = 0. \end{cases}$$

For $\hat{g}$ sufficiently small, there exists $M > 0$, such that for all $t \in [0, t')$

$$\frac{1}{p}\|y_\omega(t)\|_P^p \leq M + M\int_0^t \|y_\omega(\tau)\|^p d\tau$$
$$+ \int_0^t \|y_\omega(\tau)\|_P^{p-1} < \beta(y_\omega(\tau)), \ PCBu_\omega(\tau) > d\tau. \tag{2.9}$$

If $\varepsilon = 0$ in (1.6), then (2.9) holds for $\omega \equiv 0$.

Proof. We proceed in several steps. By Remark 2.3, we may assume that (1.11) is of the form (2.5), (2.6).

(a) We prove that, for $\hat{g} > 0$ sufficiently small, there exists $M_1 > 0$ such that, for all $t \in [0, t')$,

$$\|z_\omega(\cdot)\|_{L_p(0,t)} \leq M_1 + M_1\|y_\omega(\cdot)\|_{L_p(0,t)} + M_1\|d_\omega(\cdot)\|_{L_p(0,t)}. \tag{2.10}$$

(2.5) is equivalent to

$$z_\omega(t) = \Phi(t, 0)z_\omega(0)$$
$$+ \int_0^t \Phi(t, \tau)[A_3 y_\omega(\tau) + g_\omega^2(\tau, y_\omega(\tau), z) + d_\omega(\tau)]d\tau, \tag{2.11}$$

where $\Phi(t, \tau)$ denotes the transition matrix of

$$\dot{v}(t) = [A_4 + (\omega(t) + \dot{\omega}(t))I_{n-m}]v(t). \tag{2.12}$$

The system (2.12) is exponentially stable since $\lim_{t \to t'} \omega(t) = 0$, $\dot{\omega}(t) < 0$ and $\sigma(A_4) \subset \mathbb{C}_-$. Thus, there exist $M_2, \varepsilon > 0$, such that (2.7) and (2.11) yield

$$\|z_\omega(t)\| \leq M_2 \exp(-\varepsilon t)\|z_\omega(0)\| + \int_0^t M_2 \exp(-\varepsilon(t-\tau))[(1+\hat{g})\|y_\omega(\tau)\|$$
$$+ \hat{g}\|z_\omega(\tau)\| + \|d_\omega(\tau)\|]d\tau. \tag{2.13}$$

It follows (see e.g., Vidyasagar, 1978, p. 250) that the operator

$$\mathscr{L} : L_p(0, t') \to L_p(0, t'),$$
$$v(\cdot) \mapsto \left(t \mapsto \mathscr{L}(v)(t) := \int_0^t \exp(-\varepsilon(t-\tau))\|v(\tau)\|d\tau \right)$$

is well-defined and L_p-stable for all $p \geq 1$. In particular,

$$\|\mathscr{L}(v)(\cdot)\|_{L_p(0,t)} \leq \frac{1}{\varepsilon}\|v(\cdot)\|_{L_p(0,t)} \quad \text{for all } t \in [0, t'). \tag{2.14}$$

Taking L_p-norms in (2.13) and using (2.14) yields

$$\|z_\omega(\cdot)\|_{L_p(0,t)} \le M_2 \frac{1}{\varepsilon}\|z_\omega(0)\| + M_2(1+\hat{g})\frac{1}{\varepsilon}\|y_\omega(\cdot)\|_{L_p(0,t)}$$
$$+ M_2 \hat{g}\frac{1}{\varepsilon}\|z_\omega(\cdot)\|_{L_p(0,t)} + M_2\frac{1}{\varepsilon}\|d_\omega(\cdot)\|_{L_p(0,t)},$$

and hence, (2.10) follows for $\hat{g}$ sufficiently small.

(b) Next we show that for some $M_4 > 0$ it holds true that for all $t \in [0, t')$

$$\frac{1}{p}\|y_\omega(t)\|_P^p \le M_4 + M_4\int_0^t \|y_\omega(\tau)\|_P^p d\tau$$
$$+ M_4 \int_0^t \|y_\omega(\tau)\|_P^{p-1}[\|z_\omega(\tau)\| + \|d_\omega(\tau)\|]d\tau$$
$$+ \int_0^t \|y_\omega(\tau)\|_P^{p-1} < \beta(y_\omega(\tau)),\ PCBu_\omega(\tau) > d\tau. \qquad (2.15)$$

Note that $y(\cdot)$ is an absolutely continuous function. Let $J_1 \subset [0, \infty)$ be the set of measure zero where $y(\cdot)$ is not differentiable and

$$J_2 := \{t \in [0, \infty) \setminus J_1 \mid y(t) = 0 \text{ and } \dot{y}(t) \ne 0\}.$$

It is easy to see that $\|y(\cdot)\|_P$ is not differentiable in any point of J_2. However, $\|y(\cdot)\|_P$ is absolutely continuous because $y(\cdot)$ is and hence, J_2 must be of measure zero. It follows that $J := J_1 \cup J_2$ is of measure zero and a routine calculation gives, for all $t \in [0, \infty) \setminus J$,

$$\frac{d}{dt}\|y(t)\|_P = \begin{cases} \dfrac{< y(t),\ P\dot{y}(t) >}{\|y(t)\|_P}, & y(t) \ne 0, \\ 0, & y(t) = 0. \end{cases}$$

(2.5) together with (2.7) yields, for all $t \in [0, \infty) \setminus J$ and for some $M_5, M_6 > 0$,

$$\frac{1}{p}\frac{d}{dt}(\|y_\omega(t)\|_P^p) = \|y_\omega(t)\|_P^{p-1} < \beta(y_\omega(t)),\ P\dot{y}_\omega(t) >$$
$$\le \|y_\omega(t)\|_P^{p-1}\{< \beta(\|y_\omega(t)\|),\ M_5[\|y_\omega(t)\|_P$$
$$+ \|z_\omega(t)\| + \|d_\omega(t)\|] > + < \beta(y_\omega(t)),\ PCBu_\omega(t) >\}$$
$$\le M_6\|y_\omega(t)\|_P^p + M_5\|y_\omega(t)\|_P^{p-1}[\|z_\omega(t)\| + \|d_\omega(t)\|]$$
$$+ \|y_\omega(t)\|^{p-1} < \beta(y_\omega(t)),\ PCBu_\omega(t) >.$$

Since J is of measure zero, integrating the previous inequality yields (2.15).

(c) An application of Hölder's inequality gives for $q = p/(p-1)$, $1/q + 1/p = 1$ and every $v(\cdot) \in L_p(0, t)$

$$\int_0^t \|y_\omega(\tau)\|^{p-1}\|v(\tau)\| d\tau \le \left[\int_0^t \|y_\omega(\tau)\|^{(p-1)q} d\tau\right]^{\frac{1}{q}} \cdot \left[\int_0^t \|v(\tau)\|^p d\tau\right]^{\frac{1}{p}}$$
$$= \|y_\omega(\cdot)\|_{L_p(0,t)}^{p-1}\|v(\cdot)\|_{L_p(0,t)}. \qquad (2.16)$$

Applying this inequality to (2.15), using (2.10), and the fact that, by (1.6), $\|d_\omega(\cdot)\|_{L_p(0,t)}$ is finite since $\lim_{t\to t'} \omega(t) = 0$, yields, for suitable $M_7 > 0$,

$$\begin{aligned}\frac{1}{p}\|y_\omega(t)\|_P^p &\le M_4 + M_4\|y_\omega(\cdot)\|_{L_p(0,t)}^p \\ &\quad + M_4\|y_\omega(\cdot)\|_{L_p(0,t)}^{p-1}[\|z_\omega(\cdot)\|_{L_p(0,t)} + \|d_\omega(\cdot)\|_{L_p(0,t)}] \\ &\quad + \int_0^t \|y_\omega(\tau)\|_P^{p-1} < \beta(y_\omega(\tau)),\ PCBu_\omega(\tau) > d\tau \\ &\le M_7\,(1+\|y_\omega(\cdot)\|_{L_p(0,t)}^{p-1} + \|y_\omega(\cdot)\|_{L_p(0,t)}^p) \\ &\quad + \int_0^t \|y_\omega(\tau)\|_P^{p-1} < \beta(y_\omega(\tau)),\ PCBu_\omega(\tau) > d\tau.\end{aligned}$$

This proves the proposition.

An important application of the proposition is the following result, which demonstrates, that the exponentially weighted output $y_\omega(\cdot)$ is L_p-bounded if the system adaptive gain $k(t)$ diverges and $\omega(t)$ tends to zero as t tends to $+\infty$.

2.6 Proposition Suppose $(A,\ B,\ C)\in\Sigma$ is subject to disturbances (1.6)–(1.8), and the feedback law

$$u(t) = -\operatorname{sgn}(CB)k(t)y(t) \tag{2.17}$$

is applied to (1.11). Suppose furthermore, that $x(\cdot):[0,\ t')\to\mathbb{R}^n$ is an absolutely continuous solution of the closed-loop system with maximal interval of existence $[0,\ t')$, $t'\in(0,\ \infty]$, $\omega(\cdot):[0,\ t')\to\mathbb{R}$ is continuously differentiable and non-increasing with $\lim_{t\to t'}\omega(t)=0$, $k(\cdot):[0,\ t')\to\mathbb{R}$ is continuous and monotonically non-decreasing with $\lim_{t\to t'}k(t)=\infty$. Then, for $\hat{g}$ sufficiently small, the exponentially weighted output response of the closed-loop system (1.11), (2.17) satisfies

$$y_\omega(\cdot)\in L_p(0,\ t') \quad \text{for all} \quad p\in[1,\ \infty].$$

Proof. Let $P>0$ denote the unique solution of

$$P(CB) + (CB)^T P = \operatorname{sgn}(CB)\cdot I_n.$$

Applying (2.17) to (2.9) yields for some $M>0$, for all $p\ge 1$, and all $t\in[0,\ t')$

$$\frac{1}{p}\|y_\omega(t)\|_P^p \le M + M\int_0^t \|y_\omega(\tau)\|^p d\tau - \frac{q}{2}\int_0^t k(\tau)\|y_\omega(\tau)\|^p d\tau, \tag{2.18}$$

where $q:=(s_{\min}(P))^{(p-2)/2}$ and $s_{\min}(P)$ denotes the smallest singular value of P. Choose $\bar{t}\in(0,\ t')$ sufficiently large so that $(2/q)M<k(\bar{t})$. Then it follows from (2.18) and the monotonicity of $k(t)$, that

$$\begin{aligned}\frac{1}{p}\|y_\omega(t)\|_P^p &\le M + \int_0^{\bar{t}}\left[M - k(\tau)\frac{q}{2}\right]\|y_\omega(\tau)\|^p d\tau \\ &\quad + \left[M - k(\bar{t})\frac{q}{2}\right]\int_{\bar{t}}^t \|y_\omega(\tau)\|^p d\tau.\end{aligned} \tag{2.19}$$

If $y_\omega(\cdot)\notin L_p(0,\ t')$ for some $p\ge 1$, then the right hand side of (2.19) becomes negative, which is a contradiction. Therefore, $y_\omega(\cdot)\in L_p(0,\ t')$ for all $p\ge 1$, and (2.19) yields $y_\omega(\cdot)\in L_\infty(0,\ t')$. This completes the proof.

In contrast, if the gain remains bounded and $\omega(\cdot)$ has a finite, nonzero positive limit, then L_p-boundedness of the exponentially weighted output leads naturally to the desired exponential decay of the system state. More precisely, we have the following proposition.

2.7 Proposition Suppose $(A, B, C) \in \Sigma$ is subject to disturbances (1.6)–(1.8), and the feedback law

$$u(t) = -\operatorname{sgn}(CB)k(t)y(t) \tag{2.20}$$

is applied to (1.11). Suppose furthermore, that $x(\cdot):[0, t') \to \mathbb{R}^n$ is an absolutely continuous solution of the closed-loop system with maximal interval of existence $[0, t')$, $t' \in (0, \infty]$, $\omega(\cdot):[0, \infty) \to \mathbb{R}$ is continuously differentiable and non-increasing with $\lim_{t\to\infty} \omega(t) = \omega_\infty > 0$, and $k(\cdot):[0, \infty) \to \mathbb{R}$ is continuous and monotonically non-decreasing and bounded. Then, for $\hat{g}$ sufficiently small, the solution of the closed-loop system (1.11), (2.20) satisfies

(i) If $y(\cdot) \in L_p(0, \infty)$ for any $p \geq 1$ and $\varepsilon = 0$ in (1.7), then $\lim_{t\to\infty} x(t) = 0$.

(ii) If $y_\omega(\cdot) \in L_p(0, \infty)$ for any $p \geq 1$, then, for some $M, \omega > 0$, $\|x(t)\| \leq M\exp(-\omega t)$ for all $t \geq 0$.

Proof. We assume that (1.11) is of the form (2.5), (2.6). We only prove (ii), (i) being simpler and uses the same argument. We know that $y_\lambda(\cdot) \in L_p(0, \infty)$ for all $\lambda \in [0, \omega_\infty]$. Substitute $\omega(\cdot)$ by λ in (2.4). For $\lambda > 0$ and $\hat{g} > 0$ small enough (2.5) yields $z_\lambda(\cdot) \in L_p(0, \infty)$, this can be shown by using exactly the same arguments as in the proof of Proposition 2.5. Thus, $x_\lambda(\cdot) \in L_p(0, \infty)$. Since $k(\cdot)$ is bounded and g and h are linearly bounded, it follows from (2.5) that $\dot{x}_\lambda(\cdot) \in L_p(0, \infty)$. Now (ii) is a consequence of Lemma 2.4.

In the following sections the results will be frequently used to motivate algorithm construction and subsequently applied in the proofs of stability of the proposed adaptive control schemes.

3. Exponential Stabilization in the Case of CB of Known Sign

If the sign of CB is known then, previous work indicates that the feedback law

$$u(t) = -\operatorname{sgn}(CB)k(t)y(t) \tag{3.1}$$

is capable of ensuring asymptotic stability of the closed-loop system by suitable $k(\cdot)$ chosen adaptively. Our concern is to choose this adaptation so that the solution of the closed-loop system is also *exponentially* decaying. The method of achieving this is motivated by the following arguments.

If (A, B, C) is in the class Σ, then for $\omega > 0$ sufficiently small, $(A + \omega I_n, B, C)$ is also in Σ. If the adaptation mechanism is chosen to ensure that $x_\omega(\cdot)$ is an asymptotical stable (and hence bounded) solution of the closed-loop system

$$\dot{x}_\omega(t) = [(A + \omega I_n) - k(t)\operatorname{sgn}(CB)BC]x_\omega(t), \tag{3.2}$$

then the solution of

$$\dot{x}(t) = [A - k(t)\operatorname{sgn}(CB)BC]x(t), \tag{3.3}$$

given by $x(t) = \exp(-\omega t)x_\omega(t)$, must exponentially decay. An example of such an adaptation mechanism is the exponentially weighted controller $\dot{k}(t) = \exp(\omega t)\|y(t)\|^2$ which consequently may yield the desired stabilization result. However, it does require knowledge of a suitable value of ω. In order to apply the strategy explained above, one possible development is (see Owens et al., 1987; Logemann, 1990) to strengthen the minimum phase condition defining the system class Σ to satisfy

$$\det\begin{bmatrix} sI_n - A & B \\ -C & 0 \end{bmatrix} \neq 0 \quad \text{for all } s \in \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > -\omega\} \tag{3.4}$$

for some ***known*** $\omega > 0$. In practice, the assumption that ω is known is not expected to be realistic.

It is hence natural to consider adaptive schemes that adaptively attempt to find a suitable value of ω on-line. This basic idea was introduced for a special control law and linear systems in Ilchmann and Owens (1990), where it has been shown that exponential stabilization can be achieved by choosing ω adaptively using the control scheme defined by

$$\left.\begin{aligned} \dot{k}(t) &= \exp(2\omega(t)t)\|y(t)\|^2, \quad k(0) \geq 0 \\ \omega(t) &= \begin{cases} 1 & \text{for } t \in [0, h) \\ \dfrac{1}{1 + k(t-h)} & \text{for } t \geq h \end{cases} \end{aligned}\right\}, \tag{3.5}$$

where $h \geq 0$ is arbitrary.

The purpose of this section is to extend these results to include nonlinearities of the form discussed in Sec. 1, and more general forms of adaptation on $\omega(\cdot)$. More precisely, the following fairly rich class of adaptive functions $\omega(\cdot)$ and gains $k(\cdot)$ will be admitted.

Consider continuously differentiable functions $\omega(\cdot): [0, \infty) \to [0, \infty)$ satisfying the conditions

$$\left.\begin{aligned} &\omega(k) \text{ is non-increasing in } k \geq 0 \\ &\omega(k) > 0 \text{ for all } k \geq 0 \text{ if } \omega(\cdot) \not\equiv 0 \\ &\lim_{k \to \infty} \omega(k) = 0 \end{aligned}\right\}. \tag{3.6}$$

Let $p \geq 1$, and $K: [0, \infty) \times [0, \infty) \times \mathbb{R}^m \to \mathbb{R}$ be a Carathéodory function, so that for every integrable $y(\cdot): [0, \infty) \to \mathbb{R}^m$ the initial value problem

$$\dot{k}(t) = K(t, (\omega \circ k)(t), y(t)), \quad k(0) = k_0 \in [0, \infty) \tag{3.7}$$

has a maximal absolutely continuous solution $k(\cdot): [0, t') \to \mathbb{R}$ for some $t' > 0$, and each maximal solution satisfies the conditions

$$\left.\begin{array}{l} k(t) \geq 0 \text{ and non-decreasing in } t \\ y_{\omega\circ k}(\cdot) \in L_i(0, t') \text{ for all } i \in [p, \infty] \Rightarrow k(\cdot) \in L_\infty(0, t') \\ k(\cdot) \in L_\infty(0, t') \Rightarrow y_{\omega\circ k}(\cdot) \in L_p(0, t') \end{array}\right\} \quad (3.8)$$

(3.6) means, that we will permit any exponent that is positive and non-increasing as a function of the controller gain, and also becomes zero, if and only if the gain is unbounded.

If $h=0$, then (3.5) is an example for (ω, k) satisfying (3.6)–(3.8). We are also able to generalize (3.7) in the sense that it becomes a functional differential equation (see Hale, 1977) in order to include the case $h>0$. However, for sake of simplicity, this is omitted here.

3.1 Examples Let

$$\omega(k) = \frac{\alpha}{\beta + k^q}$$

for $\alpha, \beta > 0$, $q \geq 1$ and

$$\dot{k}(t) = \|\exp((\omega \circ k)(t))\, y(t)\|^p \cdot \sum_{i=1}^{p} F_i(y(t)), \quad k(0) \geq 0,$$

where $F_i : \mathbb{R} \to \mathbb{R}$ are polynomials, such that $F_i(\lambda) \geq F_0 > 0$ for all $\lambda \in \mathbb{R}$, $i = 1, \cdots, p$. Then $\omega(\cdot)$ satisfies (3.6), and the solution $k(\cdot)$ satisfies (3.8).

The simplest example satisfying (3.6)–(3.8) is, for arbitrary $p \geq 1$,

$$\dot{k}(t) = \|y_{\omega\circ k}(t)\|^p, \quad \omega(k) = (1+k)^{-1}, \quad k(0) \geq 0.$$

The following result is the major result of this section and states simply, that gain adaptations of a form, following from the above construction, simultaneously ensure exponential stabilization of the system state whilst guaranteeing finite limits for all adapted parameters. The system is assumed to be subject to state nonlinearities and unmodeled system dynamics but possesses no input/output nonlinearities.

3.2 Theorem Suppose $p \geq 1$, and ω, K, k satisfy (3.6)–(3.8). Let $(A, B, C) \in \Sigma$ be subject to disturbances (1.6)–(1.8) in the form (1.11). If the sign of CB is known, and if the linear bound $\hat{g}$ of $g(\cdot, \cdot)$ is sufficiently small, then the feedback law

$$u(t) = -\operatorname{sgn}(CB)\, k(t)\, y(t)$$

applied to any system of the form (1.11) yields a closed-loop system

$$\left.\begin{array}{rl} \dot{x}(t) = & [A - \operatorname{sgn}(CB)k(t)BC]x(t) + g(t, x(t)) \\ & + d(t) + Bh(t, x(t)), \quad x(0) \in \mathbb{R}^n \\ y(t) = & Cx(t) \\ \dot{k}(t) = & K(t, (\omega\circ k)(t), y(t)), \quad k(0) \geq 0 \end{array}\right\}, \quad (3.9)$$

which has a solution $(x(\cdot),\ k(\cdot)) : [0,\ t') \to \mathbb{R}^{n+1}$, for some $t' \in (0,\ \infty]$, and every solution satisfies on its maximal interval of existence $[0,\ t')$ the properties

(i) $t' = \infty$.
(ii) $\lim_{t\to\infty} k(t) = k_\infty < \infty$.
(iii) $\lim_{t\to\infty} (\omega \circ k)(t) = \omega_\infty > 0$ if $\omega(\cdot) \not\equiv 0$.
(iv) $y_{\omega\circ k}(\cdot),\ y_{\omega_\infty}(\cdot) \in L_p(0,\ \infty)$.
(v) If $\omega(\cdot) \not\equiv 0$, then there exist $M,\ \lambda > 0$, such that $\|x(t)\| \le M \exp(-\lambda t)$ for all $t \ge 0$.
(vi) If $\omega(\cdot) \equiv 0$ and $\varepsilon = 0$ in (1.6), then $\lim_{t\to\infty} x(t) = 0$.

Proof.

(a) We prove $k(\cdot) \in L_\infty(0,\ t')$. Suppose the contrary, then (3.6) yields $\lim_{t\to t'} \omega \circ k(t) = 0$. Now Proposition 2.6 yields $y_{\omega\circ k}(\cdot) \in L_p(0,\ t')$ for all $p \in [1,\ \infty]$. By (3.8) $k(\cdot)$ is bounded, which is a contradiction. Thus, $k(\cdot) \in L_\infty(0,\ t')$.

(b) Since g and h are linearly bounded on x and y, and $k(\cdot)$ is bounded on $[0,\ t')$, it follows from the theory differential equations, that $x(\cdot)$ does not have finite escape time. This proves (i).

(c) (ii) follows from (a) and (b). (iii) is a consequence of (3.6). (iv) results from (3.8), and (v) is proved by Proposition 2.7. This completes the proof.

4. Nussbaum-type Switching Strategies

In this section, the exponential stabilization results of the previous section are extended to the case of nonlinearly perturbed systems in Σ, where it is not known whether the spectrum of CB lies in the right- or left-half plane. The essential structure of the theory is similar but careful attention must be given to the definition of the choice of switching function $S(t)$. To begin the analysis, we recall the basic definition of a switching function, a concept due to Nussbaum (1983).

4.1 Definition A piecewise right continuous function $S(\cdot) : \mathbb{R} \to \mathbb{R}$ is called a *switching function* if for some $a \in \mathbb{R}$ it satisfies

$$\sup_{x>a} \frac{1}{x-a} \int_a^x S(\tau)\, d\tau = +\infty \quad \text{and} \quad \inf_{x>a} \frac{1}{x-a} \int_a^x S(\tau)\, d\tau = -\infty. \tag{4.1}$$

$N(\cdot) : \mathbb{R} \to \mathbb{R}$ is called a *Nussbaum gain* if the function $S(t) = N(t)t$ is a switching function.

A switching function $S(\cdot)$ is called *scaling-invariant* if, for all $\alpha,\ \beta > 0$, the function

$$\tilde{S}(t) := \begin{cases} \alpha \cdot S(t) & \text{if } S(t) \ge 0, \\ \beta \cdot S(t) & \text{if } S(t) < 0 \end{cases}$$

is a switching function as well. An analogous definition holds for Nussbaum functions.

4.2 Examples

(i) Nussbaum (1983) has shown that the function $N(t) = \cos(\pi/2)t \cdot \exp(t^2)$

is a Nussbaum gain. Further examples are given by $N_1(t) = \sin t$, $N_2(t) = \cos t$,

$$N_3(t) = \begin{cases} 1, & n^2 \le |t| < (n+1)^2, \quad n \text{ even}, \\ -1, & n^2 \le |t| < (n+1)^2, \quad n \text{ odd}, \end{cases}$$

$$N_4(t) = \begin{cases} 1, & 0 \le |t| < \lambda_0, \\ 1, & \lambda_n \le |t| < \lambda_{n+1}, \quad n \text{ even}, \\ -1, & \lambda_n \le |t| < \lambda_{n+1}, \quad n \text{ odd}, \end{cases}$$

where $\lambda_0 > 1$ and $\lambda_{n+1} := \lambda_n^2$.

(ii) Switching functions are for instance $S_1(t) = t \sin\sqrt{|t|}$, $S_2(t) = t\cos\sqrt{|t|}$, or $S_3(t) = \sin t \cdot t^2$.

(iii) Logemann and Owens (1988) have shown that $N(\cdot)$ is scaling-invariant, that $N_3(\cdot)$ is not scaling invariant, and that $N_4(\cdot)$ is a scaling-invariant bounded Nussbaum function.

The following remarks can be easily proven.

4.3 Remark

(i) If (4.1) is valid for some $a \in \mathbb{R}$, then it holds for all $a \in \mathbb{R}$.

(ii) (4.1) is equivalent to

$$\sup_{x>a}\left[\alpha x + \beta\int_a^x S(\tau)\,d\tau\right] = +\infty \text{ and } \inf_{x>a}\left[\alpha x + \beta\int_a^x S(\tau)\,d\tau\right] = -\infty, \tag{4.2}$$

where $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$ are arbitrary.

In the following, the adaptive feedback law is taken to have the structure

$$u(t) = -S(s(t))k(t)y(t), \tag{4.3}$$

where $S(\cdot): \mathbb{R} \to \mathbb{R}$ is a switching function driven by the solution of

$$\dot{s}(t) = k(t)\|y_{\omega\circ k}(t)\|^p, \quad s(0) \in \mathbb{R}. \tag{4.4}$$

The use of such a form, in the case $p = 2$ and $\omega = 0$, was suggested by Owens et al. (1987).

The following result proves exponential stabilization with convergence of all adaptive controller parameters for the case of systems in Σ perturbed by state space nonlinearities and unmodeled feedback dynamics, but in the absence of input and output nonlinearities.

4.4 Theorem Suppose $p \ge 1$, $S(\cdot)$ is a switching function, which, if $m > 1$, is scaling invariant, and ω, K, k satisfy (3.6)–(3.8). Let $(A, B, C) \in \Sigma$ be subject to disturbances (1.6)–(1.8) in the form (1.11). If the linear bound $\hat{g}$ of $g(\cdot,\cdot)$ is sufficiently small, then the feedback law (4.3), (4.4) applied to any system of the form (1.11) yields a closed-loop system

$$\left.\begin{aligned} \dot{x}(t) &= [A - S(s(t))k(t)BC]x(t) + g(t, x(t)) \\ &\qquad + d(t) + Bh(t, x(t)), \quad x(0) \in \mathbb{R}^n \\ y(t) &= Cx(t) \\ \dot{s}(t) &= k(t)\|y_{\omega\circ k}(t)\|^p, \qquad\qquad s(0) \in \mathbb{R} \\ \dot{k}(t) &= K(t, (\omega\circ k)(t), y(t)), \quad k(0) \geq 0 \end{aligned}\right\}, \tag{4.5}$$

which has a solution $(x(\cdot), s(\cdot), k(\cdot)):[0, \infty) \to \mathbb{R}^{n+2}$, for some $t' \in (0, \infty]$, and every solution satisfies on its maximal interval of existence $[0, t')$ the properties

(i) $t' = \infty$.
(ii) $\lim_{t\to\infty} k(t) = k_\infty < \infty$ and $\lim_{t\to\infty} s(t) = s_\infty < \infty$.
(iii) $\lim_{t\to\infty} (\omega\circ k)(t) = \omega_\infty > 0$ if $\omega(\cdot) \not\equiv 0$.
(iv) $y_{\omega\circ k}(\cdot),\ y_{\omega_\infty}(\cdot) \in L_p(0, \infty)$.
(v) If $\omega(\cdot) \not\equiv 0$, then there exist $M, \lambda > 0$, such that $\|x(t)\| \leq M\exp(-\lambda t)$ for all $t \geq 0$.
(vi) If $\omega(\cdot) \equiv 0$ and $\varepsilon = 0$ in (1.6), then $\lim_{t\to\infty} x(t) = 0$.

Proof. Existence of the solution $(x(\cdot), s(\cdot), k(\cdot)):[0, t') \to \mathbb{R}^{n+2}$, for some $t' > 0$, is a consequence of the assumption that all nonlinearities satisfy the Carathéodory condition. Let $P > 0$ denote the unique solution of

$$P(CB) + (CB)^T P = 2\operatorname{sgn}(CB) \cdot I_n.$$

Now we proceed in several steps.

(a) We prove that $k(\cdot) \in L_\infty(0, t')$. If this does not hold true, then (3.6) yields $\lim_{t\to t'} \omega\circ k(t) = 0$. Therefore, we obtain from (2.9), for all $t \in [0, t')$

$$\begin{aligned} \frac{1}{p}\|y_{\omega\circ k}(t)\|_P^p &\leq M + M\int_0^t \|y_{\omega\circ k}(s)\|_P^p\, ds \\ &\quad - \int_0^t S(s(\tau))k(\tau)\|y_{\omega\circ k}(\tau)\|_P^{p-1} < \beta(y_{\omega\circ k}(\tau)), PCBy_{\omega\circ k}(\tau) > d\tau. \end{aligned} \tag{4.6}$$

Defining

$$\tilde{S}(t) := \begin{cases} -(s_{\min}(P))^{\frac{p-2}{2}} S(t)\operatorname{sgn}(CB), & \text{if } S(t)\operatorname{sgn}(CB) \geq 0, \\ -\|P\|^{\frac{p-2}{2}} S(t)\operatorname{sgn}(CB), & \text{if } S(t)\operatorname{sgn}(CB) < 0, \end{cases}$$

where $s_{\min}(P)$ denotes the smallest singular value of P, yields

$$\begin{aligned} \frac{1}{p}\|y_{\omega\circ k}(t)\|_P^p &\leq M + M\int_0^t \|y_{\omega\circ k}(s)\|_P^p\, ds + \int_0^t \tilde{S}(s(\tau))k(\tau)\|y_{\omega\circ k}(\tau)\|^p d\tau \\ &\leq M + Mk(0)^{-1}\|P\|^p \int_0^t k(s)\|y_{\omega\circ k}(s)\|^p ds \\ &\quad + \int_{s(0)}^{s(t)} \tilde{S}(\mu)\, d\mu, \end{aligned} \tag{4.7}$$

where, without loss of generality, we have assumed $k(0) > 0$, otherwise choose $t_0 \in [0, t')$, such that $k(t_0) > 0$ and replace 0 by t_0 in the inequality.

Inserting $s(t)-s(0)=\int_0^t k(s)\|y_{\omega\circ k}(s)\|^p ds$ into (4.7), we conclude from (4.2), that the right hand side of (4.7) takes arbitrary large positive and negative values. This contradiction proves $k(\cdot)\in L_\infty(0, t')$.

(b) Now (4.7) yields $y_{\omega\circ k}(\cdot)\in L_p(0, t')$ and thus, it follows from (4.5) that $s(\cdot)\in L_\infty(0, t')$.

(c) The remainder of the proof is similar to the Proof of Theorem 3.3. We omit it.

Note that the above treatment differs from that of Willems and Byrnes (1984), who specify a switching function dependent on the gain $k(t)$ rather than our preferred choice of the derived signal $s(t)$. The above result remains valid, if $s(t)$ is replaced by $k(t)$ in the switching function to produce the control $u(t)=-S(k(t))k(t)y(t)$. The price paid for this, is that the gain adaptation must be replaced by the specific evolution $\dot{k}(t)=\|y_{\omega\circ k}(t)\|^p$ rather than the more general adaptations described in the above result. Considering now the single-input single-output case with non-linear sensor and actuator dynamics as introduced in (1.9), (1.10), we assume that $\xi(\cdot,\cdot)$ and $\eta(\cdot,\cdot)$ are linearly sector bounded in the following sense.

4.5 Definition Let $\beta\geq\alpha>0$. Then the set of *sector bounded function* $\mathcal{SB}(\alpha, \beta)$ consists of all Carathéodory functions $f(\cdot,\cdot):[0, \infty)\times\mathbb{R}$, $(t, v)\mapsto f(t, v)$ which satisfy

$$\alpha v^2 \leq f(t, v)\cdot v \leq \beta v^2 \quad \text{for almost all } t\geq 0 \text{ and all } v\in\mathbb{R}.$$

It is now possible to prove an adaptive stabilization result for single-input single-output systems in Σ perturbed by state space nonlinearities and in the presence of input and output nonlinearities of the form defined above. The scaling invariance property of $S(\cdot)$ is now needed, which is not the case, if Theorem 4.4 is restricted to single-input single-output systems.

4.6 Theorem Suppose $p\geq 1$, and ω, K, k satisfy (3.6)–(3.8), and let $S(\cdot)$ be a scaling invariant switching function. Let the single-input single-output system $(A, b, c)\in\Sigma$ be subjected to disturbances (1.6)–(1.10) in the form (1.11), where $\xi\in\mathcal{SB}(\underline{\xi}, \overline{\xi})$, $\eta\in\mathcal{SB}(\underline{\eta}, \overline{\eta})$, for some $\overline{\xi}\geq\underline{\xi}>0$, $\overline{\eta}\geq\underline{\eta}>0$. Then the feedback law

$$\left.\begin{aligned} \tilde{u}(t) &= -S(s(t))k(t)\eta(t, y(t)) \\ \dot{s}(t) &= k(t)|\eta(t, y_{\omega\circ k}(t))|^p, \quad s(0)\in\mathbb{R} \end{aligned}\right\} \tag{4.8}$$

applied to (1.11) produces a closed-loop system which possesses a solution $(x(\cdot), s(\cdot), k(\cdot)):[0, t')\to\mathbb{R}^{n+2}$, for some $t'\in(0, \infty]$, and every solution satisfies, on its maximal interval of existence $[0, t')$, the properties (i)–(vi) stated in Theorem 4.4.

Proof. Existence of the solution $(x(\cdot), s(\cdot), k(\cdot)):[0, t')\to\mathbb{R}^{n+2}$, for some $t'>0$, is a consequence of the assumption that all nonlinearities satisfy the Carathéodory condition. Suppose $k(\cdot)\notin L_\infty(0, t')$. Then, by (3.8), $s(\cdot)\notin L_\infty(0, t')$. Choose $t_0\in[0, t')$ so that $k(t_0)>0$. For almost all $t_0\in[0, t')$ we have

$$y_{\omega\circ k}(t)\, cbu_{\omega\circ k}(t) = y_{\omega\circ k}(t)\, cb \exp[(\omega\circ k)t]\xi(t, -S(s(t))k(t)\eta(t, y(t)))$$
$$\leq -cbk(t)\hat{S}(s(t))\, y_{\omega\circ k}(t)^2, \tag{4.9}$$

where

$$\hat{S}(s) := \begin{cases} \underline{\xi}\underline{\eta} S(s), & \text{if } cbS(s) \geq 0, \\ \overline{\xi}\overline{\eta} S(s), & \text{if } cbS(s) < 0 \end{cases}$$

and hence, by inequality (2.9) for $P = 1$ and 0 replaced by t_0, we have for all $t \in [t_0, t')$,

$$\frac{1}{p}|y_{\omega\circ k}(t)|^p \leq M + M\underline{\eta}^{-p}\int_{t_0}^{t} |\eta(\tau, y_{\omega\circ k}(\tau))|^p d\tau$$
$$- cb\int_{t_0}^{t} k(\tau)\hat{S}(s(\tau))|y_{\omega\circ k}(\tau)|^p d\tau$$
$$\leq M + M[\underline{\eta}k(t_0)]^{-p}\int_{t_0}^{t} k(\tau)|\eta(\tau, y_{\omega\circ k}(\tau))|^p d\tau$$
$$- cb\int_{t_0}^{t} k(\tau)\tilde{S}(s(\tau))|\eta(\tau, y_{\omega\circ k}(\tau))|^p d\tau,$$

where

$$\tilde{S}(s) := \begin{cases} \overline{\eta}^{-p}\hat{S}(s), & \text{if } cbS(s) \geq 0, \\ \underline{\eta}^{-p}\hat{S}(s), & \text{if } cbS(s) < 0. \end{cases}$$

Therefore,

$$\frac{1}{p}|y_{\omega\circ k}(t)|^p \leq M + M[\underline{\eta}k(t_0)]^{-p}[s(t) - s(t_0)] - cb\int_{s(t_0)}^{s(t)} \tilde{S}(\mu)\, d\mu,$$

and since $s(\cdot) \notin L_\infty(0, t')$, Remark 4.3 (ii) yields that the right hand side of the above inequality takes negative values, contradicting the non-negativeness of the left hand side, and hence, $s(\cdot) \in L_\infty(0, t')$. The remainder of the proof can be carried out in a similar manner as the Proof of Theorem 3.3. We omit it for brevity.

4.7 Remark If it is known that $cb > 0$ in Theorem 4.6, then the feedback mechanism can be simplified to

$$\tilde{u}(t) = -k(t)\eta(t, y(t)),$$
$$\dot{k}(t) = |\eta(t, y_{\omega\circ k}(t))|^p, \quad k(0) \in \mathbb{R}.$$

The proof of this claim is a simplification of that of Theorem 4.6.

5. Exponential Tracking

In this section, a universal exponential adaptive tracking controller is presented for the class of systems

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) \in \mathbb{R}^n \\ y(t) &= Cx(t) \\ (A,\ B,\ C) &\in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n} \\ (A,\ B,\ C) &\text{ satisfies (1.2), (1.3), and } n \text{ is arbitrary} \end{aligned}\right\}. \tag{5.1}$$

Given a class of reference signals defined by

$$\mathscr{Y}_{\text{ref}} := \{\, y_{\text{ref}} \in \mathscr{C}^\infty(\mathbb{R},\ \mathbb{R}^m) \mid \alpha\left(\frac{d}{dt}\right) y_{\text{ref}}(t) \equiv 0 \,\},$$

the objective is to construct an adaptive control law, such that for any linear system belonging to the class (5.1) and any reference signal $y_{\text{ref}}(\cdot) \in \mathscr{Y}_{\text{ref}}$, the closed-loop output response $y(t)$ generates an error $e(t) = y(t) - y_{\text{ref}}(t)$ decaying exponentially to zero. In the above, $\alpha(s) \in \mathbb{R}[s]$ is a ***known*** monic polonomial with zeros in $\overline{\mathbb{C}}_+$ only. Note that $0 \in \mathscr{Y}_{\text{ref}}$, therefore, it is not relevant to consider the case that $\alpha(s)$ has zeros in $\mathbb{C}_-$, since the corresponding modes are decaying exponentially.

The main idea, which goes back to Mareels (1984), is to use the knowledge of $\alpha(\cdot)$ to construct an internal model (that is a duplicated model of the dynamic reference signals) as part of a precompensator in the feedback loop. More precisely, let $\beta(s) \in \mathbb{R}[s]$ be a monic Hurwitz polynomial of degree $p = \deg \alpha(\cdot)$, and choose a minimal realization of $\beta(s)/\alpha(s)$ denoted by $(\hat{A},\ \hat{B},\ \hat{C},\ 1) \in \mathbb{R}^{p\times p} \times \mathbb{R}^p \times \mathbb{R}^{1\times p} \times \mathbb{R}$. Then the precompensator is given by

$$\dot{\xi}(t) = \hat{A}^* \xi(t) + \hat{B}^* v(t),\ u(t) = \hat{C}^* \xi(t) + I_m v(t),\ \xi(0) \in \mathbb{R}^m, \tag{5.2}$$

where

$$\begin{aligned} \hat{A}^* &= \operatorname{diag}\{\hat{A},\ \cdots,\ \hat{A}\} \in \mathbb{R}^{mp\times mp}, \\ \hat{B}^* &= \operatorname{diag}\{\hat{B},\ \cdots,\ \hat{B}\} \in \mathbb{R}^{mp\times m}, \\ \hat{C}^* &= \operatorname{diag}\{\hat{C},\ \cdots,\ \hat{C}\} \in \mathbb{R}^{m\times mp}. \end{aligned}$$

The internal model (5.2) is connected in series with previous adaptive stabilizers to obtain the following adaptive tracking result.

5.1 Theorem Suppose ω, K, k satisfy (3.6)–(3.8), let $S(\cdot)$ be a scaling invariant switching function, and let $(\hat{A}^*,\ \hat{B}^*,\ \hat{C}^*)$ be as in (5.2). Then for arbitrary initial conditions $\xi(0) \in \mathbb{R}^{mp}$, $k(0) \in \mathbb{R}$, and reference signal $y_{\text{ref}}(\cdot) \in \mathscr{Y}_{\text{ref}}$, the following error feedback controller:

$$\left.\begin{aligned} e(t) &= y(t) - y_{\text{ref}}(t) \\ \dot{k}(t) &= K(t,\ \omega \circ k(t),\ e(t)), \quad k(0) \in \mathbb{R} \\ v(t) &= -S(k(t))k(t)e(t) \\ \dot{\xi}(t) &= \hat{A}^* \xi(t) + \hat{B}^* v(t), \quad \xi(0) \in \mathbb{R}^{mp} \\ u(t) &= \hat{C}^* \xi(t) + I_m v(t) \end{aligned}\right\} \tag{5.3}$$

applied to any system belonging to the class (5.1), with arbitrary initial condition $x(0) \in \mathbb{R}^n$, yields a unique solution $(x(\cdot), \xi(\cdot), k(\cdot)) : [0, t') \to \mathbb{R}^{n+mp+1}$ with the properties

(i) $t' = \infty$.
(ii) $\lim_{t\to\infty} k(t) = k_\infty < \infty$.
(iii) $\lim_{t\to\infty} (\omega \circ k)(t) = \omega_\infty > 0$ if $\omega(\cdot) \not\equiv 0$.
(iv) If $\omega(\cdot) \not\equiv 0$ then there exists $M, \lambda > 0$, such that $\|e(t)\| \le M \exp(-\lambda t)$ for all $t \ge 0$.
(v) If $\omega(\cdot) \equiv 0$, then $\lim_{t\to\infty} e(t) = 0$.
(vi) There exists a $c > 0$, such that for all $t \ge 0$.

$$\|(\xi(t), x(t))\| \le c\,(1 + \max_{s\in[0,t]} \{\|y_{\mathrm{ref}}(s)\|\}).$$

Proof. The input-output behavior $v \mapsto y$ of the series interconnection formed by (5.2) and (5.1) is described by

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}v(t), \; y(t) = \bar{C}\bar{x}(t), \; \bar{x}(0) \in \mathbb{R}^{n+mp}, \tag{5.4}$$

where

$$\bar{A} = \begin{bmatrix} A & B\hat{C}^* \\ 0 & \hat{A}^* \end{bmatrix}, \; \bar{B} = \begin{bmatrix} B \\ \hat{B}^* \end{bmatrix}, \; \bar{C} = [C, \; 0], \; \bar{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}.$$

By construction, the transfer matrix $\bar{G}(s) = \bar{C}(sI_{n+mp} - \bar{A})^{-1}\bar{B}$ has no zeros in $\bar{\mathbb{C}}_+$. It is easy to see that (5.4) is stabilizable and detectable, thus, by Proposition 2.1, minimum phase.

The essential ingredients of the present proof is the following lemma proved by Miller and Davison (1991) (see also Townley and Owens, 1991).

Lemma. For every $y_{\mathrm{ref}}(\cdot) \in \mathscr{Y}_{\mathrm{ref}}$, there exists a $x_0 \in \mathbb{R}^{n+mp}$, such that

$$y_{\mathrm{ref}}(t) = \bar{C}\tilde{x}(t), \quad \dot{\tilde{x}}(t) = \bar{A}\tilde{x}(t), \quad \tilde{x}(0) = \tilde{x}_0, \tag{5.5}$$

$$\|\tilde{x}(t)\| \le c\left(1 + \max_{\tau\in[0,t]} \{\|y_{\mathrm{ref}}(\tau)\|\}\right) \quad \text{for all } t \ge 0. \tag{5.6}$$

By (5.4) and (5.5), $x_e(t) := \bar{x}(t) - \tilde{x}(t)$ satisfies

$$\dot{\bar{x}}_e(t) = \bar{A}\bar{x}_e(t) + \bar{B}v(t), \; \bar{x}_e(0) = \bar{x}(0) - \tilde{x}(0),$$

$$e(t) = \bar{C}x_e(t).$$

The problem has hence been converted into a standard stabilization problem. Since $(\bar{A}, \bar{B}, \bar{C})$ is minimum phase and $\bar{C}\bar{B} = CB$, Theorem 4.4 can be applied and $x_e(t)$ is decaying exponentially (resp. asymptotically) for $\omega \not\equiv 0$ (resp. $\omega \equiv 0$). This proves (i)–(v). (vi) is a consequence of $\lim_{t\to\infty} x_e(t) = 0$ and of (5.6). This completes the proof.

5.2 Remark Theorem 5.1 has been proved for *asymptotic* tracking and very special gain adaptations $\dot{k} = K(e)$, for single-input single-output systems (see by Mareels, 1984; Helmke et al., 1990), for multi-input multi-output systems (see Miller and Davison (1991) and a less general version in Townley and Owens (1991)), and for infinite dimensional, multi-input multi-output systems (see by Logemann and Ilchmann, 1991).

5.3 Remark The adaptive tracking controller (5.3) cannot, in general, tolerate nonlinearities $g(\cdot,\cdot)$, $h(\cdot,\cdot)$ in the nominal system (5.1), if $\mathscr{Y}_{\text{ref}} \neq \{0\}$. This is not due to *exponential* tracking, but to the fact, that, if the gain converges, the terminal system is a nonlinear system. It cannot be expected, in general, that a reference signal produced by a linear system can be tracked by such a nonlinear system.

6. Example

Let $n = 2$ and $m = 1$ and consider the following system:

$$\left.\begin{aligned} \dot{x}(t) &= Ax(t) + g(t, x(t)) + d(t) + Bu(t) \\ y(t) &= Cx(t) \\ u(t) &= \xi(t, \tilde{u}(t)) \\ \tilde{y}(t) &= \eta(t, y(t)) \end{aligned}\right\} \tag{6.1}$$

with

$$A = \begin{bmatrix} 2 & -2 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$C = [1, 0], \quad x(0) = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

and

$$\begin{aligned} d(t) &= \exp\left(-\frac{t}{5}\right)\cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ g(x) &= 0.5\begin{bmatrix} x_2\cos(x_1) \\ x_1\sin(x_2) \end{bmatrix}, \\ \xi(\tilde{u}) &= (2+\cos(\tilde{u}))\tilde{u}, \quad \eta(y) = (2+\sin(y))\,y. \end{aligned}$$

Obviously, (A, B, C) is minimum phase, $g(\cdot,\cdot)$ is norm bounded, $d(\cdot) \in L_p(0, \infty)$, and $\xi(\cdot,\cdot)$, $\eta(\cdot,\cdot)$ are sector bounded. Assuming that the sign of the high-frequency gain cb is known, we choose, according to Remark 4.7, the adaptation mechanism

$$\left.\begin{aligned} \tilde{u}(t) &= -k(t)\eta(y(t)) \\ \dot{k}(t) &= \exp\left(\frac{2}{1+k(t)}\right)\eta(y(t))^2, \quad k(0) = 0 \end{aligned}\right\}. \tag{6.2}$$

The output, sector bounded nonlinearly perturbed output, and the gain evolution of the feedback system (6.1), (6.2) is shown in Fig. 2.

If the sign of the frequency gain is unknown, we choose, according to Theorem 4.7, the feedback strategy

$$\left.\begin{aligned} \tilde{u}(t) &= S(s(t))k(t)\eta(y(t)) \\ \dot{k}(t) &= \exp\left(\frac{2}{1+k(t)}\right)\eta(y(t))^2, \quad k(0) = 0 \\ \dot{s}(t) &= \exp\left(\frac{2}{1+k(t)}\right)\eta(y(t))^2, \quad s(0) = 0 \\ S(\lambda) &= -\cos(2.2\lambda)\exp(0.001\lambda^2) \end{aligned}\right\}. \tag{6.3}$$

The dynamics of the closed-loop system (6.1), (6.3) are illustrated in Fig. 3.

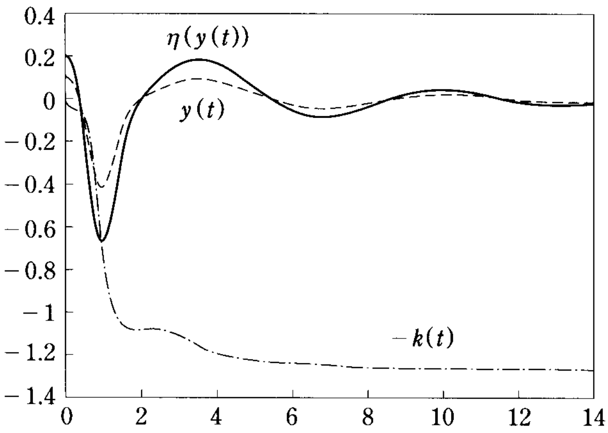

Fig. 2. Output and gain evolution for known high-frequency gain.

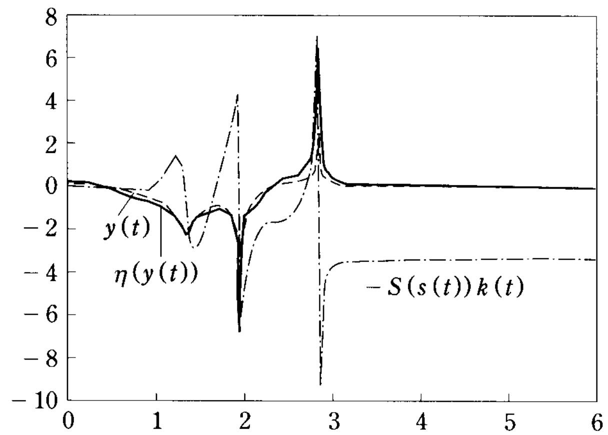

Fig. 3. Output and gain evolution for unknown high-frequency gain.

7. Conclusions

The paper has successfully extended current algorithms for universal adaptive stabilization and tracking to guarantee exponential decay of the systems state in the presence of a well-defined class of disturbances and of state space and input-output nonlinear perturbations of an underlying linear model. The linear model is assumed to be time-invariant, to have relative degree one, and to be minimum phase, but otherwise is arbitrary. The mechanism for achieving the results is adaptive exponential weighting of systems output in the gain evolution laws and switching functions, and the derivation of an integral inequality relating exponentially weighted inputs and outputs. An additional innovation introduced in the paper is the replacement of previously used L_2-based analysis by an L_p-based analysis. This has benefits in extending the class of gain adaptations considerably and removes the need for essentially L_2-based adaptation mechanisms required in other works. An extension of the present results to minimum phase systems which satisfy $\det(CB) \neq 0$ only is the topic of future research.

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